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Partial Differential Equations - Final Exam

You have 3 hours to complete this exam. Please show all work. Each question has point values listed next to it for a total of 100 points. Be sure to quote clearly any theorems you use from the textbook or class. Good luck!

1. (20 points) Consider the problem

$$u''(x) + u'(x) = f(x) \tag{0.1}$$

$$u'(0) = u(0) = \frac{1}{2}\left(u'(l) + u(l)\right) \tag{0.2}$$

with $f(x)$ a given function
(10 points) a) Is the solution unique? Justify your answer.
(10 points) b) Does a solution exist, or is there a condition that $f(x)$ must satisfy for existence? Justify your answer.

2. (20 points total) (10 points) a) The functions $f_n(x) = \frac{x^2}{n} + x$ as $n \to \infty$, do they converge pointwise to x? Do they converge uniformly? Justify your answers
(10 points) b) Prove that if $f \in C^2(\mathbb{R})$ and periodic with period 2π, then f has a Fourier series which converges uniformly to f.

3. (30 points total) (15 pts) a) Use the method of separation of variables to solve the wave equation:

$$\frac{\partial^2 \phi}{\partial t^2}(x,t) = \frac{\partial^2 \phi}{\partial x^2}(x,t) \tag{0.3}$$

for the function with boundary conditions

$$\phi(0,t) = 0 \quad \text{and} \quad \phi(\pi,t) = 0 \tag{0.4}$$

and the initial conditions

$$\phi(x,0) = \sin 3x \quad \text{and} \quad \frac{\partial \phi}{\partial t}(x,0) = 0 \tag{0.5}$$

(5 points) b) Solve instead with boundary conditions

$$\phi(0,t) = 0 \quad \text{and} \quad \phi(\pi,t) = \pi \tag{0.6}$$

and the initial conditions

$$\phi(x,0) = \sin 3x + \pi x \tag{0.7}$$

(10 points) c) The general solution is of the form:

$$f(x+t) + g(x-t) \tag{0.8}$$

Rewrite the solutions in part b) in this form. Determine f and g. *Hint, you may wish to use the identity:*

$$2 \sin A \cos B = \sin(A-B) + \sin(A+B) \tag{0.9}$$

4. (20 points) Let Ω be an open, bounded, connected subset of $\mathbb{R}^d$ and let $\phi \in C^2(\Omega) \cap C(\overline{\Omega})$ be *subharmonic* in Ω, that is

$$-\Delta\phi \leq 0 \tag{0.10}$$

Prove the weak maximum principle that:

$$\max_{x\in\overline{\Omega}} \phi(x) = \max_{x\in\partial\Omega} \phi(x) \tag{0.11}$$

(Hint: Establish the result for the special case where $-\Delta\phi < 0$ and then consider the function $\psi(x) = \phi(x) + \epsilon e^{x_1}$).
What can you say about *super harmonic* ($-\Delta\phi \geq 0$) and *harmonic* ($-\Delta\phi = 0$) functions?

5. (10 points) Solve the equation

$$au_x + bu_y + cu = 0 \tag{0.12}$$

where a, b, c are constants, and $u = u(x, y)$.